JOURNAL OF APPROXIMATION THEORY 1, 391–400 (1968)

# Continuity of the Best Approximation Operator for Restricted Range Approximations

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## 1. INTRODUCTION

In this paper we examine the behavior of the best uniform rational approximation operator in certain generalized weight function approximation problems. An introduction to this subject is given in [*2*].

Let $X$ be a compact topological space, and for $f \in C(X)$ let

$$\|f\| = \max_{x \in X} |f(x)|.$$

Let $P$ and $Q$ be two finite dimensional linear subspaces of $C(X)$. In generalized rational approximation one is interested in approximating an $f \in C(X)$ by a function of the form $r = p/q$ where $p \in P$, $q \in Q$ and $q > 0$ on $X$.

A generalized weight function $W(x, y)$ is defined for $x \in X$, $y$ real, and has values in the extended reals. Specific examples and a number of results concerning generalized weight functions are given in ([*1*], [*2*], [*3*], [*4*]). In this paper we are concerned with the problem of finding a generalized rational function $r$ which minimizes

$$\sup_{x \in X} |W[x, f(x) - r(x)]|. \tag{1}$$

The sections which follow give a number of results concerning (1), assuming various hypotheses on $W(x, y)$ and on the space of functions $P + rQ$ where $r$ is a solution to the approximation problem. Here $P + rQ = \{p + rq : p \in P, q \in Q\}$.

Certain notations are used throughout the paper. Suppose for a fixed rational function $r$ that $P + rQ$ has a basis $g_1, \ldots, g_n$. Then for $x \in X$ we define a vector $\hat{x}$ by

$$\hat{x} \equiv (g_1(x), g_2(x), \ldots, g_n(x)). \tag{2}$$

The symbol 0 denotes the origin of Euclidean $n$-space. Suppose $Y$ is a subset of $X$, and $g$ is a real valued function defined on $Y$. Then

$$H\{g(y)\,\hat{y} : y \in Y\}$$

denotes the convex hull of the set of vectors $g(y)\,\hat{y}$ with $y \in Y$.

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[1] Supported by N.S.F. Grant GP-8686.

If $G$ is a linear subspace of $C(X)$, of dimension $k$, then $G$ is called a Haar subspace iff every nonzero element of $G$ has at most $k-1$ zeros.

## 2. Restricted Range Approximations

Let $l$ and $u$ be two elements of $C(X)$ satisfying

$$l(x) < u(x) \ \forall \ x \in X.$$

Let $f^* \in C(X)$ be the function to be approximated, and define

$$R = \{r \equiv p/q : p \in P, q \in Q, q > 0, l \leqslant f^* - r \leqslant u\}. \tag{3}$$

In the discussion which follows we always assume that $R$ is nonempty.

We shall consider a generalized weight function $W(x, y)$ with the following properties:

If $D = \{(x, y): x \in X, y \text{ real}, l(x) \leqslant y \leqslant u(x)\}$ then:

$$\left.\begin{array}{ll} \text{(a)} & W(x, y) \text{ is continuous over } D; \\ \text{(b)} & \partial W(x, y)/\partial y \text{ is continuous over } D \text{ and positive at each point} \\ & (x, y) \text{ of } D \text{ with } y \neq 0; \\ \text{(c)} & (x, y) \in D \Rightarrow \operatorname{sgn} W(x, y) = \operatorname{sgn} y; \\ \text{(d)} & x \in X \text{ and } y > u(x) \Rightarrow W(x, y) = \infty; \\ \text{(e)} & x \in X \text{ and } y < l(x) \Rightarrow W(x, y) = -\infty. \end{array}\right\} \tag{4}$$

These hypotheses are satisfied, for example, in the problem considered in [4].

For notational convenience we write

$$E(f^* - r)(x) \equiv W[x, f^*(x) - r(x)].$$

We call $E(f^* - r)$ the weighted error function. Thus the problem (1) is to minimize

$$\sup_x |E(f^* - r)(x)| \equiv \|E(f^* - r)\|.$$

In restricted range approximations there are two types of critical points. For a particular $r \in R$ under consideration define:

$$X_{+1} = \{x \in X: E(f^* - r)(x) = \|E(f^* - r)\|\}$$

$$X_{-1} = \{x \in X: E(f^* - r)(x) = -\|E(f^* - r)\|\}$$

$$X_{+2} = \{x \in X: E(f^* - r)(x) = u(x)\}$$

$$X_{-2} = \{x \in X: E(f^* - r)(x) = l(x)\}$$

$$X_r = X_{+1} \cup X_{-1} \cup X_{+2} \cup X_{-2}.$$

In [1] it was shown that the cases $X_{+1} \cap X_{-2} \neq \varnothing$ and $X_{-1} \cap X_{+2} \neq \varnothing$ are exceptional, and not of general interest. Here we shall assume

$$X_{+1} \cap X_{-2} = X_{-1} \cap X_{+2} = \varnothing .$$

Then if $f^* \not\equiv r$ we can define an integer valued function $\sigma_r$ on $X_r$ as follows

$$\sigma_r(x) = \begin{cases} \operatorname{sgn} E(f^* - r)(x) & x \in X_{+1} \cup X_{-1} \\ +1 & x \in X_{+2} \\ -1 & x \in X_{-2}. \end{cases}$$

For the remainder of this section we assume $f^* \notin R$. The following characterization theorem and lemma, which we shall need later, are established in [1].

THEOREM 1. *If $P + rQ$ is a Haar subspace then $r$ is a best approximation to* $f^*$ iff

$$0 \in H\{\sigma_r(x)\,\hat{x}\colon x \in X_r\}.$$

LEMMA 1. *If $P + rQ$ is a Haar subspace then*

$$0 \in H\{\sigma_r(x)\,\hat{x}\colon x \in X_r\}$$

iff *there is no nonzero $h \in P + rQ$ such that $(\sigma_r h)(x) \geqslant 0$ for all $x \in X_r$.*

If $r^*$ is a best approximation to $f^*$ from $R$ and $P + r^* Q$ is a Haar subspace, then $r^*$ is unique [1]. In this situation we shall denote $r^*$ by $\tau f^*$. We shall establish the continuity of the operator $\tau$ at a normal point $f^* \in C(X)$.

DEFINITION. $f^* \in C(X)$ is a normal point iff it has a best approximation $r^*$ from $R$ such that $P + r^* Q$ is a Haar subspace whose dimension = dimension $P$ + dimension $Q - 1$.

Results concerning normal points can be found in ([5], [6], [7]). The first result we shall prove here is a *strong uniqueness theorem.*

THEOREM 2. *Let $r^*$ be a best approximation to $f^*$ from $R$. If $f^*$ is normal then there exists an $\alpha > 0$ such that for all $r \in R$*

$$\|E(f^* - r)\| \geqslant \|E(f^* - r^*)\| + \alpha\|E(f^* - r^*) - E(f^* - r)\|. \tag{5}$$

*Proof.* (Note that this result is trivially true if $f^* \in R$.) We assume $f^* \not\equiv r^*$ and that there is no $\alpha$ as stated. Then there exist sequences $\{r_n\} \subset R$ and $\{\alpha_n\}$, where $\alpha_n \to 0$ and

$$\alpha_n\|E(f^* - r^*) - E(f^* - r_n)\| = \|E(f^* - r_n)\| - \|E(f^* - r^*)\|.$$

Here $r_n = p_n/q_n$, $q_n > 0$, $\|p_n\| + \|q_n\| = 1$, and $r_n \not\equiv r^*$. Since $l \leqslant f^* - r_n \leqslant u$, $\{r_n\}$ is bounded. Here there is no loss of generality in assuming that there exist $p \in P, q \in Q$ such that $\|p\| + \|q\| = 1$ and $p_n \to p$, $q_n \to q$. We also can assume $r^* = p^*/q^*$ where $\|p^*\| + \|q^*\| = 1$. For simplicity of notation we shall write $\sigma(x) \equiv \sigma_{r^*}(x)$.

If $x \in X_{+1} \cup X_{-1}$ then

$$\begin{aligned}\alpha_n \|E(f^* - r^*) - E(f^* - r_n)\| \\ &= \|E(f^* - r_n)\| - \|E(f^* - r^*)\| \\ &\geqslant \sigma(x)\{W[x, f^*(x) - r_n(x)] - W[x, f^*(x) - r^*(x)]\} \\ &= \sigma(x) \frac{\partial W[x, y_n(x)]}{\partial y} [r^*(x) - r_n(x)]. \qquad (6)\end{aligned}$$

Here $y_n(x)$ is between $f^*(x) - r_n(x)$ and $f^*(x) - r^*(x)$. For the fixed $x$ under consideration it might happen that zero is a point of accumulation of $\{f^*(x) - r_n(x)\}$. If that happens then by choosing subsequences one can assume $f^*(x) - r_n(x) \to 0$. Then for sufficiently large $n$,

$$\sigma(x)[r^*(x) - r_n(x)] = \sigma(x)[r^*(x) - f^*(x) + f^*(x) - r_n(x)] \leqslant 0. \qquad (7)$$

This uses the fact that

$$\sigma(x)[f^*(x) - r^*(x)] = \|(f^* - r^*)\| > 0.$$

Now by multiplying each side of (7) by $q_n(x)$ and taking limits, one concludes

$$0 \geqslant \sigma(x)[r^*(x)q(x) - p(x)]. \qquad (8)$$

If $\{f^*(x) - r_n(x)\}$ does not have zero as a point of accumulation then there exists an $N$ such that

$$d(x) \equiv \inf_{n \geqslant N} \frac{\partial W[x, y_n(x)]}{\partial y} > 0.$$

Hence for sufficiently large $n$ it follows from (6) that

$$\frac{\alpha_n}{d(x)} \|E(f^* - r^*) - E(f^* - r_n)\| \geqslant \sigma(x)[r^*(x) - r_n(x)]. \qquad (9)$$

Then by multiplying by $q_n(x)$ and taking limits one again obtains the inequality (8). That is, (8) holds for all $x \in X_{+1} \cup X_{-1}$.

For $x \in X_{+2} \cup X_{-2}$,

$$\sigma(x)[f^*(x) - r^*(x)] \geqslant \sigma(x)[f^*(x) - r_n(x)].$$

Hence

$$\sigma(x)[-r^*(x)q_n(x) + p_n(x)] \geqslant 0. \qquad (10)$$

Taking limits we again conclude that (8) holds.

Since (8) holds for all $x \in X_r$ we obtain, using Lemma 1, $-r^*q + p \equiv 0$.

It then follows from ([5], p. 165) that $p^* \equiv p$, $q^* \equiv q$, and hence $r_n \to r^*$. We conclude that zero is not an accumulation point of $\{f(x) - r_n(x)\}$ when $x \in X_{+1} \cup X_{-1}$. Thus, since in any event $r_n \to r^*$ uniformly, there is no loss of generality in assuming there exists a $d > 0$ such that for all $n$ and all $x \in X_{+1} \cup X_{-1}$,

$$d \leqslant \frac{\partial W[x, y_n(x)]}{\partial y}.$$

Since $q_n \to q^*$ uniformly, there exists a $\delta > 0$ such that for all $n$ and all $x \in X$, $q_n(x) \geqslant \delta$. By a straightforward argument, using Lemma 1 and (10), it follows that there exists a $c > 0$ such that for all $n$,

$$c \leqslant \max_{x \in X_{+1} \cup X_{-1}} \frac{\sigma(x)\,[r^*(x)\,q_n(x) - p_n(x)]}{\|r^* q_n - p_n\|}.$$

Using the above results in (6), we conclude

$$\begin{aligned} \alpha_n \|E(f^* - r^*) - E(f^* - r_n)\| &\geqslant dc\|r^* q_n - p_n\| \\ &\geqslant dc\delta\|r^* - r_n\|. \end{aligned}$$

An application of the mean value theorem to this inequality gives the existence of an $m > 0$ such that

$$m\alpha_n\|r_n - r^*\| \geqslant dc\delta\|r^* - r_n\|.$$

Since $r_n \not\equiv r^*$ and $\alpha_n \to 0$, this yields the desired contradiction and completes the proof.

We now focus our attention on the continuity of $\tau$ at a normal point $f^*$. Let

$$F = \{f \in C(X) \colon l \leqslant f - \tau f^* \leqslant u\}. \tag{11}$$

For each $f \in F$, we consider the question of finding a solution to the problem of minimizing $\|E(f - r)\|$ for $r \in R$.

THEOREM 3. *Let $f^*$ be a normal point of $C(X)$. Then there exists an $\alpha > 0$ such that $f_0 \in F$ and $\|f^* - f_0\| < \alpha$ imply that $f_0$ has at least one best approximation. Moreover, there exists a constant $\beta > 0$ such that for any best approximation $r_0$ to $f_0$,*

$$\|E(f^* - \tau f^*) - E(f_0 - r_0)\| \leqslant \beta\|(f^* - f_0)\|. \tag{12}$$

*Proof.* Let $r^*$ be the best approximation to $f^*$. The search for a best approximation to $f_0$ may be confined to those $r_0 \in R$ for which

$$\|E(f_0 - r_0)\| \leqslant \|E(f_0 - r^*)\|.$$

Such $r_0$ satisfy (using the triangle inequality)

$$\|E(f^* - r^*) - E(f_0 - r_0)\| \leqslant \|E(f^* - r^*) - E(f^* - r_0)\| \\ + \|E(f^* - r_0) - E(f_0 - r_0)\|.$$

Using Theorem 2 and then the triangle inequality and other manipulations, it follows that the above is

$$\leqslant \frac{1}{\alpha}[\|E(f^* - r_0)\| - \|E(f^* - r^*)\|] + \|E(f^* - r_0) - E(f_0 - r_0)\|$$

$$\leqslant \frac{1}{\alpha}[\|E(f^* - r_0) - E(f_0 - r_0)\| + \|E(f_0 - r_0)\| - \|E(f^* - r^*)\|] \\ + \|E(f^* - r_0) - E(f_0 - r_0)\|$$

$$\leqslant \frac{1}{\alpha}[\|E(f^* - r_0) - E(f_0 - r_0)\| + \|E(f_0 - r^*)\| - \|E(f^* - r^*)\|] \\ + \|E(f^* - r_0) - E(f_0 - r_0)\|$$

$$\leqslant \frac{1}{\alpha}[\|E(f^* - r_0) - E(f_0 - r_0)\| + \|E(f_0 - r^*) - E(f^* - r^*)\|] \\ + \|E(f^* - r_0) - E(f_0 - r_0)\|.$$

Application of the mean value theorem to each of the three "normed" quantities above, leads to the result (12). The proof is then completed by use of the methods in [*5*], p. 168, and [*6*].

It is worth noting that many generalized weight function approximations which do not have the restricted range condition can be considered to have it. For example, suppose $W(x, y)$ satisfies:

(a) $\operatorname{sgn} W(x, y) = \operatorname{sgn} y$;

(b) $W(x, y)$ and $\partial W(x, y)/\partial y$ are continuous;

(c) $\partial W(x, y)/\partial y > 0$ when $y \neq 0$, and $\lim_{|y| \to \infty} |W(x, y)| = \infty$.

This allows us to select $u(x)$ sufficiently large, and $l(x)$ sufficiently small, so that $X_{+2} = \varnothing$ and $X_{-2} = \varnothing$. Then the results of Theorems 2 and 3 hold. These results are, thus, important if one is considering the computational aspects of this problem.

Next we consider the case where $P + (\tau f^*) Q$ is a Haar subspace but $f^*$ is not necessarily a normal point of $C(X)$.

THEOREM 4. *Let* $\{f_n\} \subset F$ *and* $\{r_n\} \subset R$ *be two sequences such that*

$$f_n \to f^*$$

*and*

$$\|E(f_n - r_n)\| \to \|E(f^* - r^*)\|.$$

*Here* $r^* = \tau f^*$. *If* $r_n$ *is written in the normalized form* $r_n = p_n/q_n$, *with* $\|p_n\| + \|q_n\| = 1$, *then the sequence* $\{(p_n, q_n)\}$ *converges to the subspace*

$$M \equiv \{(p,q): p \in P, q \in Q, -p + r^* q \equiv 0\};$$

*that is,*

$$\text{distance}\,[M,(p_n,q_n)] \to 0.$$

*Proof.* If $r^* \equiv f^*$ we find $E(f_n - r_n) \to 0$, and hence by the properties of the weight function, $f_n - r_n \to 0$. Thus *a fortiori* we obtain the desired result.

If $f^* \not\equiv r^*$ and the result is false then there exist subsequences of $\{f_n\}$ and $\{r_n\}$ which we do not relabel satisfying

(a) there exist an $\epsilon > 0$ such that distance $[M,(p_n,q_n)] \geqslant \epsilon$ for all $n$;

(b) $p_n \to p$, $q_n \to q$ where $\|p\| + \|q\| = 1$.

For $x \in X_{+1} \cup X_{-1}$,

$$\|E(f_n - r_n)\| - \|E(f^* - r^*)\| \geqslant \sigma_{r^*}(x)\,[E(f_n - r_n)(x) - E(f^* - r^*)(x)].$$

Using the same techniques as were employed in the proof of Theorem 2, one can verify that

$$0 \geqslant \sigma_{r^*}(x)\,[r^*(x)q(x) - p(x)]. \tag{13}$$

Since inequality (13) also holds for $x \in X_{+2} \cup X_{-2}$, it follows by Lemma 1 that

$$r^* q - p \equiv 0.$$

This contradicts the assumption that

$$\text{distance}\,[M,(p_n,q_n)] \geqslant \epsilon \qquad \text{for all } n$$

and completes the proof.

For the remainder of this section we specialize to the situation where $X = [a,b]$. We make the assumption that for each nonzero $q \in Q$, the set of zeros of $q$ is of measure zero.

THEOREM 5. *If* $\{r_n\} \subset R$ *and* $\{f_n\} \subset F$ *are such that* $r_n = p_n/q_n$, $\|p_n\| + \|q_n\| = 1$, $(p_n,q_n) \to M$, *and* $f_n \to f^*$, *then* $E(f_n - r_n) \to E(f^* - r^*)$ *in measure. Here* $M = \{(p,q): p \in P, q \in Q, -p + r^* q \equiv 0\}$.

*Proof.* Assume the contrary. We can then find subsequences of $\{r_n\}$ and $\{f_n\}$, which we do not relabel, such that

(a) There exist an $\epsilon > 0$ and a positive integer $k$ such that if

$$B_n \equiv \{x: |E(f_n - r_n)(x) - E(f^* - r^*)(x)| > 1/k\}$$

then the measure of $B_n$ is greater than $\epsilon$ for all $n$;

(b) $p_n \to p$, $q_n \to q$ where $\|p\| + \|q\| = 1$.

Since $\|p\| + \|q\| = 1$ and $-p + r^*q \equiv 0$, we conclude that $q \not\equiv 0$. Let

$$X_0 = \{x : q(x) \neq 0\}.$$

By hypothesis the measure of $X_0$ is $b - a$. Choose a closed set $X_1 \subset X_0$ such that the measure of $X_1$ is $b - a$. On $X_1$, $E(f_n - r_n) \to E(f^* - r^*)$ uniformly. Thus for large $n$, $B_n \cap X_1 = \varnothing$, which implies that $B_n$ has measure zero. This is a contradiction.

The following result is then clear.

THEOREM 6. *If $r^*$ is a best approximation to $f^*$ and $P + r^* Q$ is a Haar subspace, then for every pair of sequences $\{r_n\} \subset R$ and $\{f_n\} \subset F$ such that $f_n \to f$ and $\|E(f_n - r_n)\| \to \|E(f^* - r^*)\|$, $E(f_n - r_n) \to E(f^* - r^*)$ in measure.*

## 3. RATIONAL APPROXIMATION WITH INTERPOLATION

We turn now to a different sort of restricted range approximation. Using the ordinary uniform norm as a measure of error we are interested in finding a best rational approximation which interpolates $f(x)$ on a prescribed point set. To be more specific, let $\{x_1, \ldots, x_k\} \subset X$, where $k \leqslant$ dimension $P$, be a given set of points. For $f \in C(X)$ let

$$R_f = \{r \equiv p/q : p \in P; q \in Q; q > 0; r(x_i) = f(x_i), \quad i = 1, \ldots, k\}$$

Then we call $r^* \in R_f$ a best approximation to $f$ from $R_f$ iff

$$\text{distance}\,(R_f, f) = \|f - r^*\|.$$

For each $r \in R_f$ define

$$S_r = \{-p + rq : p \in P; q \in Q; (-p + rq)(x_i) = 0, \quad i = 1, \ldots, k\}.$$

DEFINITION. $S_r$ is called an interpolating Haar subspace iff every nonzero element in $S_r$ has at most $d(r) - 1$ zeros distinct from $\{x_1, \ldots, x_k\}$. $d(r)$ is the dimension of the subspace $S_r$.

Clearly if $P + rQ$ is a Haar subspace, then $S_r$ is an interpolating Haar subspace. The following theorem and lemma are given in [*8*].

THEOREM 7. *$r$ is a best approximation to $f$ from $R_f$* iff

$$0 \in H\{\sigma(x)\hat{x} : x \in X_r\}$$

*where*

$$\sigma(x) = \text{sgn}\,[f(x) - r(x)],\ X_r = \{x \in X : |f(x) - r(x)| = \|f - r\|\},$$

*Here $\hat{x} \equiv (g_1(x), g_2(x), \ldots, g_n(x)$, where $g_1, g_2, \ldots, g_n$ is a basis of $S_r$.*

LEMMA 2. *If $r$ is a best approximation to $f$ from $R_f$, where $r \not\equiv f$ and $S_r$ is an interpolating Haar subspace, then $h \in S_r$ and $\sigma(x)h(x) \geqslant 0$ for all $x \in X_r$ imply $h \equiv 0$.*

In [8], under the assumption that the dimension of the interpolating Haar subspace $S_r$ is (dimension $P$ + dimension $Q - 1 - k$), the Lipschitz continuity of the best approximation operator at $f$ was demonstrated. In general we will show that only convergence in measure can be expected.

THEOREM 8. *Let $r$ be a best approximation to $f$ from $R_f$ and assume $S_r$ is an interpolating Haar subspace. Let $\{r_n\}$ and $\{f_n\}$ be two sequences with the properties:*

(a) $r_n \in R_{f_n}$, *where* $r_n = p_n/q_n$ *and* $\|p_n\| + \|q_n\| = 1$;
(b) $f_n \to f$;
(c) $\|f_n - r_n\| \to \|f - r\|$.

*Define* $M = \{(p, q) \in P \times Q : -p + rq \equiv 0\}$. *Then*

$$\text{distance } [(p_n, q_n), M] \to 0$$

*Proof.* For the case $r \equiv f$, the result is clear. If $r \not\equiv f$, assume that the result is false. Then (by taking subsequences if necessary) there exists an $\epsilon > 0$ such that

$$\text{distance } ((p_n, q_n), M) \geqslant \epsilon \tag{14}$$

for all $n$. By taking further subsequences we can secure that $p_n \to p$ and $q_n \to q$. Now, for each interpolating point $x_i$,

$$-p_n(x_i) + q_n(x_i) f_n(x_i) = 0.$$

Since $f_n(x_i) = r_n(x_i)$, one finds by taking the limit,

$$-p(x_i) + q(x_i) r(x_i) = 0.$$

Hence $-p + rq \in S_r$. By the same argument used in Theorem 2,

$$-p(x) + q(x) r(x) = 0$$

for each $x \in X_r$. Hence by Lemma 2

$$-p + rq \equiv 0.$$

This contradicts (14).

THEOREM 9. *If $r^* \in R_f$, and $P + r^* Q$ is a Haar subspace, then there exists a $\gamma > 0$ such that $\|f - g\| < \gamma$ implies that $R_g$ is nonempty. Furthermore, if $f_n \to f$ and $\|f - f_n\| < \gamma$, there exist $r_n \in R_{f_n}$ such that $r_n \to r^*$.*

*Proof.* Consider the system of equations for $p$ and $q$

$$-p(x_i) + g(x_i)q(x_i) = 0 \qquad i = 1, \ldots, k.$$

By hypothesis, this system can be solved in a neighborhood of $p = p^*$, $q = q^*$ and $g = f$ for a $p$ and $q$ such that if $r = p/q$, $r \in R_g$ and $r$ is close to $r^*$.

COROLLARY. *Under the same hypotheses as in the previous theorem,* $f_n \to f$ *implies distance* $(R_{f_n}, f_n) \to$ *distance* $(R_f, f)$.

Now if we specialize to the case where $X = [a, b]$ and assume for each nonzero $q \in Q$, that the set of zeros of $q$ has measure zero, we find, pursuing the same ideas as in the restricted range case, that:

THEOREM 10. *Assume $r$ is a best approximation to $f$ from $R_f$ and $S_r$ is an interpolating Haar subspace. Then if $\{r_n\}$ and $\{f_n\}$ are two sequences such that $f_n \to f$, $r_n \in R_{f_n}$ and $\| f_n - r_n \| \to \| f - r \|$, then $r_n \to r$ in measure.*

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